

## Comments on the Stability Criterion of Lienard and Chipart

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### ABSTRACT

Some comments are made on the stability criterion of Lienard and Chipart for the zeros of a polynomial to have negative real parts.

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The statement of Theorem 1 of the paper [1] by B. N. Datta is not quite precisely the stability criterion of Lienard and Chipart [2, p. 319], which may be stated as

**THEOREM 1'.** *Necessary and sufficient conditions for all the zeros of  $a(x)$  to have negative real part are that the Bezout matrix  $B_{h,g}$  is positive definite and that about half of the  $a_i$ 's in proper order are negative.*

The above theorem contains about  $n$  conditions for the  $n$ th-degree polynomial  $a(x)$ . It seems that Dr. Datta misquoted the original criterion of Lienard and Chipart. By doing this, he obtained  $3n/2$  conditions. However, Krein and Naimark (Reference [9] of the paper) quoted the original criterion correctly.

The original criterion of Lienard and Chipart was misquoted in the same way by Anderson [3, p. 699]. Anderson then gave a new proof of a correct form of the criterion, with about  $n$  conditions, which he attributed to Gantmacher [4, p. 221].

To prove Theorem 1' stated above, one can follow the proof of Anderson [3] by noting that for  $n$  odd,

$$B_{h,g} = JDJ,$$

where  $J$  is the matrix with ones on its secondary diagonal and zeros elsewhere, and  $D$  is a symmetric matrix defined by Anderson; and for  $n$  even,

$$B_{h, g} = JCJ,$$

where  $C$  is again a symmetric matrix defined by Anderson.

#### REMARKS.

(1) The theorem of Lienard and Chipart was proven several times (sometimes rediscovered, as in Fuller [5]) by many authors, such as Marden [6], Barnett [7] and, recently, Husseyin and Jury [8], using the inners approach.

(2) When half of the  $a_i$ 's in special order are negative and  $B_{h, g}$  is positive definite, then the other half of the  $a_i$ 's are necessarily negative. Hence, whether there are  $n$  or  $n/2$  coefficients to be examined is more an esthetic than an immediate practical consideration. Furthermore, the proof of Theorem 1 of Datta might be simpler than for the above Theorem 1'. The possibility of such a simpler proof was also mentioned by Anderson in his conclusion.

(3) It is pertinent to mention the works of Parks [9] in connection with the proof of stability criteria using Lyapunov's theorem of stability.

(4) It is of historical interest to note that Lienard [10] in an earlier paper was concerned with the redundancy of the stability conditions. This concern finally led him, in collaboration with Chipart, to publish their famous criterion. This historical fact has induced this writer to make the above comments to clear up the redundancy conditions quoted above.

(5) In Anderson and Jury [11], a conjecture was proposed that the Lienard-Chipart criterion gives the simplest set of conditions in terms of number of inequalities and total degree. However, a rigorous proof is still lacking.

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